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THE
MATHEMATICAL GAZETTE.

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ON VON STAUDT'S GEOMETRIE DER LAGE.

It is far too much the custom now to rely on the analogy of algebra to justify the introduction of imaginaries into geometry. Analogy, however, is no justification unless we first prove the exact correspondence of the fields of investigation. In analytical geometry the identification of the two fields is permissible, and is easily explained; but in pure geometry any reference to algebra, expressed or implied, is irrelevant and misleading. The elements of pure geometry have no dependence on calculation.

There is not precisely the same difficulty about the elements at infinity. They are easier of comprehension, geometrically; we feel that at any rate we know *where* they are—where we should encounter them, if we could only get there—but we can form no conception of position for imaginaries. The only attempt at a strictly geometrical theory is to be found in von Staudt's *Geometrie der Lage* (1847), and *Beiträge zur Geometrie der Lage* (1856); and later writers are mostly contented to leave it there, with a passing reference to it as supplying ample justification for the use of imaginary elements. This is unfortunate, for while the theory itself presents no special difficulties except in one or two points of the argument, yet von Staudt's presentation is at times obscure, at times tedious; and as he lacks the conciliatory style that does so much to encourage a student, very few have the patience to persevere to the end. In these pages I propose to give a brief account of von Staudt's system of geometry, premising however that owing to the small amount of explanation that he vouchsafes, I may at times be reading my own interpretation into his text.

The proper introduction of infinitely distant and imaginary elements into the domain of geometry depends, naturally, on the way in which the material is defined, the whole development of the system being but a series of deductions from the definitions.

In a strictly formal geometry all the material may be regarded as given, absolutely, with the laws of combination; the difference between real and imaginary elements, or between elements at a finite distance and elements at infinity, will make itself felt only when we attempt to compare the given material with the visible universe. The difficulty is then to determine what part of the intellectual domain can be represented by visible realities. But von Staudt's procedure is altogether different. His elements are in the first instance abstractions from the visible universe, but he enlarges the field thus given by adjoining to it other purely intellectual elements, given by formal definition. The crux here is to prove that these formal elements are of the same nature as the original elements.

Thus von Staudt ascends from a visible, actual domain to a more extensive intellectual domain. His work falls naturally into three principal divisions:—

(i.) *intuitive geometry*, dealing with elements regarded as visible entities; this includes the usual propositions of projective geometry, and other investigations such as are now classed as topological (*Geometrie*, pp. 1-23, 30-178; *Beiträge*, pp. 1-76);

(ii.) *constructive geometry*, in which he creates his own enlarged universe by formal definition (*G.* 23-30; *B.* 76-126);

(iii.) *formal geometry*, in which he considers all figures resulting from the combinations of the observed and defined elements (*B.* 131-256).

The pages here named, with those containing certain metric supplements (*G.* 203-216; *B.* 126-129, 256-283), though only a part of the 600 pages of von Staudt's two works, contain all that is necessary for the present purpose. The division by pages is not very exact; (i.) and (ii.), as also (ii.) and (iii.), overlap to a great extent.

On the whole, I do not consider the intuitive part of von Staudt's geometry synthetic; when it is so, it is rather by an afterthought. The material and its laws of combination are derived from observation, and are afterwards extended and restated in an intellectual synthetic form. His view in the first instance is this. We have before us the visible universe; this, in so far as it is visible, we will treat geometrically. But we cannot handle any part of it, and so we cannot measure anything; all we can do is to change our point of view. The objects we see (or might see, for this is what he means by *denkbar*) are bounded portions of space; from these we arrive at the idea of surfaces, lines, and finally points, each as a boundary; regarded as a boundary, each has two sides, one towards the contained figure, one towards the rest of the field (space, surface, or line) from which it is cut. The objects of our researches are therefore solids, surfaces, lines, and points, and all visible relations of these

are matters for investigation. Each of these elements can move in the preceding one, so generating it; and checking its motion at any stage, it becomes again a boundary. When, for example, a point moves in a line *ABC*, seeing that as regards the line the point has only two sides, it can move only in the two opposed senses, *ABC*, *CBA*. Two points thus determine a bounded portion of the line; if the line be not closed, one point can separate two, but if the line be closed, it takes two points to separate two given points *A*, *B*, for we can pass from *A* to *B* along the line in either sense.

Among lines and surfaces are particularly to be distinguished straight lines and planes. A straight line is a line which is fixed in space by two of its points; hence two straight lines cannot intersect in more than one point. A surface which can be generated by the motion of a straight line is said to be ruled. The straight line being of all lines the one that is most simply determined, is regarded as one of the fundamental elements of the geometry; when looked upon as a particular line described by the motion of a point it is called a straight line, but when looked upon as an element, it is called a ray (*Strahl*). Starting from any point on a ray, we distinguish two opposed, or complementary, half-rays; all half-rays proceeding from any point in space form a sheaf of half-rays (*Halbstrahlenbündel*). A part of a sheaf is a solid angle or angular space, and the surface by which the part of the sheaf is separated from the rest is a simple conical surface. Thus if a surface is such that it can be filled with half-rays of a sheaf, it is a conical surface, and it bounds an angular space. Such a surface can be described by a half-ray moving about the vertex as a pivot, and in any of its positions this half-ray serves as one boundary of a portion of the conical surface; any line joining the vertex to one other point of a conical surface lies entirely on the surface. The complementary half-rays describe the vertical angular space, with the vertical conical surface to bound it. If these vertical surfaces are two distinct surfaces, so that a ray starting fron any position and moving on one conical surface can never reverse itself, the surface is of even order; but if the vertical surfaces are one and the same, so that a half-ray can move on the surface till it coincides with its complement, thus reversing itself, the surface is of odd order. The two parts of a conical surface of even order form the complete conical surface; and if these two parts are connected only at the vertex, so that they have not a single half-ray in common, the surface divides the sheaf into two parts, of which one is a complete cone; this lies on the inside of the conical surface. The remainder of the sheaf, outside the conical surface, is such that conical surfaces of odd order can exist in it; this is the definition of "outside the surface." The complete conical surface divides the

sheaf of half-rays into three parts, of which two are simple cones. All this is of use in considering the circuits of plane curves.

A plane is a conical surface of odd order in which any point may be regarded as the vertex. Hence any line that has two points in common with the plane lies entirely in it, for one of the two points may be regarded as the vertex. From this all the ordinary propositions about planes and lines follow, as for instance that a plane and a line can cut only in one point.

It is to be noticed that von Staudt lays particular stress on the divisibility of every kind of field that presents itself; and that every element presents itself as a field, inasmuch as a system of elements of a different kind can be united with (pass through, or lie in) it. Thus in a plane, a point determines a field whose elements are straight lines, namely, the flat pencil; and in space, a line determines a field with, *e.g.* planes for elements; this is an axial pencil. The bounded portions of these two fields are called angles, plane angle and dihedral angle, determined respectively by two half-rays and two half-planes. If the half-rays or half-planes complementary to the boundary ones are included in the angle considered, it is said to be *erhaben*; if they are not included, the angle is *hohl*. Thus two half-rays divide a flat pencil into two angles, one *hohl* and one *erhaben*. If, however, the two boundary half-rays or half-planes are complementary, the angle is *flach* or straight. Thus angles less than, equal to, or greater than, two right angles are respectively *hohl*, *flach*, or *erhaben*.

A straight line, regarded as a field of points (range), has some characteristics in common with the flat pencil and the axial pencil. For joining any external point (or line) to all the points of the range, we obtain a flat pencil (or axial pencil). In each field two elements bound a portion; and an element describing the field can move only in the two opposed senses. These three fields are the primitive forms of the first grade. The primitive forms of the second grade are the plane, regarded as a field of points and lines, thus containing ranges and flat pencils, and the sheaf, that is, the point regarded as a field of planes and lines, containing therefore axial and flat pencils. The primitive form of the third grade is common space, which contains all the others.

Our only concern is with the appearance of figures; figures that appear the same are for our purposes identical. If the various points of two figures are impressed on the eye by the same rays, then these figures are in appearance, *dem Scheine nach*, identical. The set of rays by which either figure is projected to the eye is therefore called its *Schein*; figures that yield the same *Schein* are identical in appearance. Of course if we change the point of view, the figures are no longer identical *dem Scheine nach*. The

word, as von Staudt uses it, seems untranslateable; mathematically projector (used, for example, in Holgate's translation of Reye's *Geometrie der Lage*), answers well enough, but it misses the significant reference to the appearance, to the visual foundation of von Staudt's geometry. Yet *Schein* cannot be translated by any term that implies any attribute of the figure itself, nor even by aspect, for it is rather the medium, the vehicle, by which the aspect is made known to the eye.

So far as this geometry concerns itself, figures that yield the same *Schein* have the same properties; on this is based the theory of projection. Cutting, for example, the *Schein* of a plane figure by a different plane, we obtain a different figure of (*dem Scheine nach*) identical properties; this projective identity of two forms is shown to depend on harmonic equivalence.

Distinct in conception from the elements by which any given field can be generated, though identical with them in fact, are certain figures in the field, which are themselves aggregates of elements. These figures are

(1) in space, a surface;

(2) in a plane, or on any surface, a curve;

(3) on any line, a *Wurf* (cast or throw). This is the simplest set of elements about which anything can be predicated. Now as any three points on a line can be projected into any other three, a set of three points has no characteristic properties; but a set of four points has properties that are projectively indestructible, and this set is a *Wurf*. Just as curve means neither length nor area, nor even shape, though the figure has shape and can be regarded as determining a length or an area, so *Wurf* means simply the set of four points in a determinate order. The same term is applied to a set of four elements in any other primitive form of the first grade.

It is shown that forms can be projectively equivalent without being in such a position that their projective relation can be immediately discovered by the eye; it may however be indirectly tested by comparison with a third form. When two projective forms of the same nature are thus placed, the pairs of corresponding elements determine a set of new elements; for example, two ranges give rise to a set of lines, two flat pencils in the same plane yield a set of points. In this way we obtain, in a plane, a locus of the second order and an envelope of the second class; in a sheaf, a conical surface as a locus of rays or an envelope of planes; these are regarded simply as figures. But two ranges whose bases are not coplanar, or two axial pencils whose bases are not concurrent, give rise to a system of straight lines, a regulus (*Regelschaar*), which is ranked as one of the primitive forms rather than as a figure, though it is taken also as indicating a figure, namely, the ruled surface on which all the rays lie;

every plane section of this surface is either a proper conic or a pair of straight lines.

I suppose the reason for thus distinguishing the regulus is that while the primitive forms of the first grade originally given allow for every possible position of two points or two planes, since these are necessarily elements in a range or axial pencil, they do not allow for every possible relative position of two straight lines. If the lines intersect, they are components of a flat pencil; but if they do not intersect—which is, after all, the most general case—they cannot be placed in any of the primitive forms first mentioned. The regulus fills up this gap; and the primitive forms of the first grade may now be enumerated as follows:

Range of points, Axial pencil, Flat pencil, and Regulus.

Pure geometry deals with the relations of points, lines, and planes; a very small part deals with figures composed of a finite number of these; when the figures are to be continuous, the elements must be indefinite in number, and then they are handled by means of these primitive forms of the first grade.

It was no part of von Staudt's design to say in what other systems of geometry some of the lines of investigation had been started, or when and how the results were originally discovered. His business was to apply his method in all suitable regions. Thus in addition to the usual projective geometry, we find Plücker's conception of the dual generation of curves (and surfaces), with the production of true singularities by stationary elements[1]—a conception that has no dependence on any analytical statement of geometrical facts. We find also (pp. 81-110) an investigation in the theory of circuits and regions, which, together with the contemporary memoirs of Listing and Möbius,[2] may be regarded as the foundation of the modern treatment of topology.

Among the abstractions from observation on which von Staudt bases the intuitive part of his geometry are to be noticed (1) the idea of direction, (2) the axiom on which his treatment of parallels is founded. This axiom is that a straight line cannot lie entirely within a hollow angle (*hohl*). From this it follows that the *Schein* of a line with reference to any external point is a straight angle. It cannot be *erhaben*, for as all the half-rays of the *Schein* are obtained by joining the points of the line singly to the vertex, two such half-rays cannot be complementary, as the straight line thus formed would then meet the given line in two points, one for each half-ray. By the axiom, it cannot be *hohl*; hence it is

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[1] Plücker, *System der analytischen Geometrie*, p. 241 (1835); *Theorie der algebraischen Curven*, p. 200 (1839); von Staudt, *Geometrie*, pp. 110-118 (1847).

[2] Listing, *Vorstudien zur Topologie*, 1847; Möbius, *Grundformen der planen Linien dritter Ordnung*, 1849, *Selbstanzeige*, 1848.

flach or straight. Now the boundary half-rays are not a part of the angle, therefore they do not meet the given line; and as these half-rays, bounding the straight angle, form one straight line through the vertex, this is a straight line that does not meet the given line. But all other lines through the vertex do meet it, for each consists of two half-rays, of which one is an element of the *Schein* of the given line. Hence every flat pencil that lies in a plane with a given straight line contains precisely one ray that does not meet the line.

Two lines in a plane are said to be parallel if they do not meet; and what we have just proved is that through one point one line can be drawn parallel to a given line.

The direction, *Richtung*, of a line is referred to without a word of comment, it being taken for granted that parallel lines have the same direction. Any figure formed of lines (plane or sheaf) is said to contain a specified direction if it contains one line in that direction. Planes are said to be parallel if they have no point in common; it is a matter of observation that parallel planes have something in common, and this is called their aspect, *Stellung*. Two directions determine the aspect; one point and the aspect determine a plane; two aspects determine a direction. It is noticed that in the various statements direction corresponds to point, aspect to line; two lines in a plane have either a common point or a common direction; a plane contains either a point of a line or the direction of the line; two planes have in common a line, or else their aspect, and so on. A mode of referring to *Richtung* and *Stellung* that shall emphasize this likeness is desirable.

Consider the intersection of two lines in a plane; keep one line fixed, and let the other revolve about a point not on the first line. So long as the two lines have different directions, they have a determinate point of intersection; as the second line approaches the first in direction, this common point recedes indefinitely, and finally loses itself. If we now speak of the intersection of the two lines as a point at infinity, an ideal point, we express this phenomenon in symbolical language, and we replace the term *Richtung* by a phrase that exhibits the analogy between direction and point. But this amounts to extending the domain of our investigation by adjoining to it all ideal points, namely, one on every set of parallel lines: we enquire therefore whether this change of base will necessitate any alteration in the geometry we have already constructed.

The ideal point on any line is to be found in either direction along that line; hence putting on the ideal point has changed the straight line from an open line to a closed one. This statement sums up the changes caused.

If we accept these ideal points, all that lie in any one plane

must be spoken of as lying in a straight line, for there is to be one such point on any arbitrary line; we shall have therefore in each plane one ideal line, and as this is determined by two ideal points, it is fitted to replace the idea of the aspect of a plane. By adjoining the ideal line we change the plane from an open surface to a closed one. All these ideal points and lines must be regarded as lying in one ideal plane, in order that there may be one ideal point on every line, one ideal line in every plane.

These considerations lead to the conclusion that the system of geometry will be most harmoniously developed if we throw aside the restriction of our domain to the visible universe. The region of our researches is now to include the visible universe, and as much more as can be formulated and is found convenient. For the present we adjoin simply one plane; the domain is the visible universe increased by one ideal plane.

To show that this ideal plane with its elements is of the same nature as any other plane with its elements, von Staudt examines and compares the constructions and proofs for the different possible combinations of actual and ideal elements. This involves some tedious detail; but it appears unavoidable with this method of handling the subject. The conclusion to which this minute investigation leads is that the geometry of this enlarged domain is the same as that of the restricted domain—the visible universe —without the exceptions; the laws of combination of the elements now hold universally.

We can picture this ideal plane on any arbitrary plane, P_1, of the visible universe (for example, by representing its points by the points in which its *Schein* from any vertex is cut by P_1); the plane P_1 must then be represented on some other plane P_2, P_2 on P_3, and so on; there will always be one plane left over, one plane not pictured on the planes of the universe; the domain is "the universe + one plane." Von Staudt does not state this explicitly, but his attitude towards imaginary elements suggests that something of this nature may have been in his mind.

C. A. SCOTT.

(*To be continued.*)

REVIEWS.

Leçons nouvelles sur les applications géométriques du calcul différentiel. Par W. DE TANNENBERG, Professeur à la Faculté des Sciences de l'Université de Bordeaux. Paris, Hermann, 1899; 8vo, 192 pp.

Opinions seem to differ as to how far the geometrical applications of the Calculus ought to be collected from the four corners of mathematics and presented to the student *en bloc*. Many people think it is a mistake to let them occupy so much space as is generally granted to them at present in the text-books. "If I were writing a book on the Differential

Calculus," said a well-known Cambridge man to the writer some time ago, "I would spend less time in juggling with curves; it smothers what really needs to be taught."

Most teachers will perhaps agree that the pedal-of-the-evolute-of-the-inverse and other remote relatives of a given curve get more attention than they deserve from text-book writers and Tripos examiners; one is tempted to wonder whether they are as important as some of the matters which are left out, "on account," as we learn from the text-book preface, "of the limitations of expense and space."

Once admitted to be unsatisfactory, the chapters devoted to geometrical applications in the books may be altered in one of two ways: either by shortening them until their length corresponds more nearly to their importance, or by substituting fresh matter for that at present given. Many considerations seem to us to point to the former of these plans as the best. In the first place, the geometrical applications of analysis take up at present more than their fair share of a student's mathematical course. He is expected to be familiar with the most trivial properties of the conic sections and the cycloids, and to work out difficult examples on curvature, evolutes, pedals, intrinsic equations, and parabolic asymptotes.

No doubt these are interesting in their way, but they are not on the high road to anywhere of importance, and they take time; and we cannot help feeling that some retrenchment in this region would be abundantly justified if it gave the student opportunity at a later period of his course to read more about infinite series, differential equations, harmonic analysis, and theory of functions. At present a low wrangler knows next to nothing about these latter subjects.

This position may be supported by considerations of a more pedagogical character. If mathematics is to be partitioned into "subjects," to each of which is to be devoted a text-book, a course of lectures, and an examination paper, then it is undesirable that one subject should grow by swallowing up parts of others: and we think that few people can handle a text-book like that of Edwards without feeling that this has happened to the Differential Calculus.

We have thought it well to bring up this question, because the author of the book under review—De Tannenberg's *Leçons Nouvelles sur les applications géométriques du calcul différentiel*—evidently takes a different view of the matter. He, in fact, proposes to add 190 pages of fresh applications to the heavy burden we already have to bear.

The field he has chosen is the geometry of three dimensions, which is left almost untouched in the usual English text-books on the Calculus. His work, in fact, is what we generally call the Theory of Surfaces; it developes the ideas to which most students are first introduced in the concluding chapters of Smith's *Solid Geometry*, and which they possibly continue in the treatises of Salmon and Frost. But the great book to which it must be referred is Darboux's *Theory of Surfaces*, to which the author is evidently considerably indebted. And it seems to us that the best way to treat it is first to enter an objection against its title, and then to estimate its value as a text-book on the Theory of Surfaces.

The book is philosophically divided into five parts, dealing respectively with the descriptive properties of twisted curves, the descriptive properties of surfaces, the metric properties of twisted curves, the metric properties of ruled surfaces, and the metric properties of surfaces in general. And here let it be said that partitions based on such principles as the distinction between descriptive and metric properties are in our opinion really useful, and ought to appear more prominently in elementary works than they do at present.

The first two parts, which deal with such matters as envelopes and osculating planes, do not call for much notice; the last chapter of the second part, however, in a few articles on complexes and congruences, introduces the reader to the elements of line-geometry.

In the third part, which is devoted to the metric properties of twisted curves, M. de Tannenberg's favourite method is a use of the formulae connected with the "Trihedron of Serret," *i.e.*, the system of moving axes constituted by the tangent, principal normal, and binormal, at a variable point of the curve.

The fourth part consists of two chapters, one devoted to skew and one to developable surfaces; the chief results of the theory of ruled surfaces are neatly and clearly proved, and applied to problems such as that of Bertrand, "To find the conditions which must be satisfied by a curve whose principal normals are also the principal normals of another curve."

The fifth part, which is concerned with the metric properties of surfaces in general, is much longer than the others, and occupies nearly half the book. A surface S is supposed to be defined by three equations, expressing the coordinates x, y, z, in terms of two variable parameters u, v; as u and v vary, the point x, y, z, traces out the surface. Any element of arc ds, traced on the surface, can be expressed in terms of the corresponding increments of the superficial coordinates u and v by an equation of the form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where E, F, G, are functions of u and v; and these functions E, F, G, are invariant when the surface S is displaced in any manner in space. If now the direction-cosines of the normal to the surface be denoted by a, b, c, it is found that the differential form $\Sigma dadx$, or

$$-(E'du^2 + 2F'dudv + G'dv^2)$$

is likewise invariant. It is then shown that the six functions E, F, G, E', F', G', characterize the surface; in other words, if these functions are given, the surface is completely determinate, independently of its position in space. On this as basis, the author considers lines of curvature, geodesics, asymptotic lines, and the general theory of curves traced on a surface.

It is a matter for wonder that the results and methods of the Theory of Surfaces have not met with greater popularity in this country. They are interesting, divisible into pretty fragments, and more exciting than the plane geometry over which we lavish so much care. Perhaps M. de Tannenberg's book will bring this branch of mathematics into more general cultivation. If so, we shall rejoice. E. T. WHITTAKER.

Cours de Problèmes de Géométrie Analytique (Deux et Trois Dimensions). By Prof. G. DE LONGCHAMPS, ancien Professeur au Lycée Saint-Louis (3 vols. pp. 295, 435, 582, Delagrave).

The Elements of Coordinate Geometry. Part II. The Conic. By J. H. GRACE, B.A., and F. ROSENBERG, M.A. (pp. 312, W. B. Clive).

PROF. G. B. Elliott, in his stimulating address to the London Mathematical Society (1898), on "Some Secondary Needs and Opportunities of English Mathematicians," suggested that "unambitious work of definitely educational intention on the part of those who have received the higher enlightenment, is what is needed." And again, "we want to lead the young tyro where there is work to be done, to guide him sufficiently far along toilsome routes which pioneer mathematicians have traversed, that he may find new fields from which to extract the fruitfulness, and, it may be, new paths along which to pass in future discovery."

No one who is interested in mathematics can fail cordially to endorse these remarks, referring as they do to the higher branches of the subject. But they may indirectly raise in the mind of an experienced teacher a doubt as to whether the course of study indicated in the average text-book is necessarily the best to arouse and excite in the mind of the student a genuine interest, an interest likely to persist long after the claims of the examination room have been satisfied. We are proud, and rightly proud, of the skill and ingenuity which year after year constructs hundreds of original problems for the purpose of testing the powers of the student in dealing with the details of his subject. Many, however, may feel more than a passing doubt, that although facility in treating a series of "twenty minutes problems" may arise from mathematical ability of a high order, and disclose the "potential mathematician," yet the enormous amount of time that is devoted to this efficiency, and to this efficiency alone, from school life upwards, does perhaps tend to cramp and fetter other faculties, which if diverted in another direction might ultimately be productive of original work. The power of concentrating the attention upon a given point with sustained vigour for the space of twenty minutes is, as Todhunter used to say, the object of our training. But that power, valuable as it is, by no means implies the power of spending days, weeks, or months in paths which may lead to a "future discovery." The whole question would seem to reduce to:—are we to consider mathematics in the school and the university as a mere discipline, or as an end? If it be considered as an end, then there would seem to be something wrong in our system of training. How many of the young men who leave the university with an Honours degree take enough interest in the subject that has occupied three years of their young manhood to make any attempt at original work? How many of their names are to be found attached even to problems or solutions in the *Educ. Times Reprint*, or similar periodicals? Not one per cent. Three years of problem grinding would seem to have destroyed all interest in problems. What if it has also destroyed all interest in mathematics! In the French schools and universities the same difficulty is felt. Professor de Longchamps, who has watched the evil grow for the last quarter of a century, sets the ideal

very clearly in his preface to the last of the three volumes, which form as it were his legacy to the overburdened students of the lycées and kindred institutions. "*Le but véritable n'est-il pas de faire d'un élève sortant de nos grandes écoles, un esprit apte à tout comprendre? un esprit resté ambitieux et bien vivant, ayant le goût de la science et des recherches si variées qu'elle présente aux intelligences avisées et curieuses?*" And again, the result upon students, "*surchauffés, déjà fatigués, . . . epuisés, désabusés, dégoûtés à l'avance de toutes les recherches personnelles,*" he well describes: *toujours penser par le Maître, jamais ou trop rarement par lui-même, voir tous les jours grandir les programmes dont l'étude absorbe tout son temps, ne lui laissant aucun répit pour se reprendre et se féconder, par l'effort intérieur, la pensée qui n'a fait que l'effleurer; tel est, si j'ai bien observé, . . . le sort de l'Etudiant en Mathématiques.*" And if the result be to produce students, "dégoûtés à l'avance de toutes les recherches personelles," the charge is conceived in language none too vigorous.

A peculiar interest would therefore seem to attach to the *Cours de Problèmes* which has been prepared by this veteran who so sorrowfully plays the Cassandra over the future of mathematics in France. Compelled to follow the rigid outline of a State programme, he has endeavoured, and with remarkable success, to establish a *via media* in the domain of the Geometry of Two and Three Dimensions, between the unfertile methods, the barrenness of which he bewails, and the inspiring methods, of which, in these volumes, he has shewn himself a distinguished exponent.

To say that no teacher or mathematical enthusiast should be without this book is no random assertion. The lover of problems will find food for his pastime to last him for many months. Very few of the questions are to be found elsewhere, and, with rare self-denial, the author has deliberately abstained from utilising any of the large number of questions he has proposed during the last quarter of a century in the columns of the various mathematical periodicals. Every result has been verified, many by several methods, geometrical as well as analytical. The teacher will learn more of the technique of his profession, from the lucidity of presentation, the precision of statement, and the ability with which the subject matter is coordinated, than from a wilderness of ordinary text-books. The enthusiast will realize the advantage of reading a text-book written by one who has experienced the "higher enlightenment," which, perhaps, will be most apparent in the continual instances of "generalization." But the essential merit of these volumes may best be summed up in the simple phrase of a mathematician of European eminence: *un livre excitateur.*

Messrs. Grace and Rosenberg bring us on to another plane. They are of course heavily handicapped by having to write a book not merely for a specific examination, the London B.A., but we imagine, especially for the private student. Of its kind it is excellent; that is to say that every point which is likely to give the student a momentary pause is fully elucidated, *e.g.* there are *four* chapters on the Tracing of Conics! There is the usual apparatus of leaded lines, illustrative examples, and graduated problem papers. The authors have closely followed Salmon in the order of treatment, a step on which, in spite of its disadvantages, they are to be congratulated.

The attitude of the authors to the syllabus of the London B.A., is, we note with satisfaction, resentful; "any book confined to that syllabus" would be "very incomplete." But that is not the worst thing that may befall a book written for a syllabus. In justice, however, we must say that this work is, for its purpose, the best, save one, of the series in which it is the latest volume. Most of us will appreciate the humour of a remark in the preface :—"Many (of the more difficult exercises) are original, which, of course, generally means that the plagiarism is unconscious." Is this a milder form of the ancient "pereant illi qui ante nos nostra dixerunt?"

W. J. GREENSTREET.

MATHEMATICAL NOTES.

75. *On the Quadrilaterals connected with four coplanar forces in equilibrium.*

If three forces are in equilibrium, their lines of action must meet in a point. If four forces are in equilibrium this is no longer necessary, but there is an interesting relation (for coplanar forces) between the force polygon and the quadrilateral formed by the lines of action of the forces in question, the forces being taken in the same order for each. For equilibrium, *the quadrilaterals, though not in general similar, must have their diagonals as well as their sides parallel.* For let the four forces be represented in magnitude and direction by $A'B'$, $B'C'$, $C'D'$, $D'A'$; and let them act along the lines AB, BC, CD, DA respectively. The two first forces will have a resultant represented by $A'C'$ acting at B; the two last a resultant represented by $C'A'$ acting at D. For equilibrium BD must be parallel to $A'C'$: similarly AC must be parallel to $B'D'$. Thus the quadrilaterals have their non-corresponding diagonals parallel, instead of (as in the case of similar figures similarly placed) those joining corresponding angles.

As it is obviously sufficient for equilibrium that BD should be parallel to $A'C'$, we infer the geometrical proposition that if quadrilaterals, with their sides respectively parallel, have one pair of diagonals parallel, the other pair must be parallel likewise; and that not only when the diagonals which are parallel correspond (when the result is obvious from the fact that the quadrilaterals will be similar), but also in the opposite case.

This geometrical result is most easily verified by constructing the force polygon so as to be adjacent to the other quadrilaterals, as $EFCD$ in the figure, where EF is parallel to AB. Then $ABDFEC$ will be a hexagon inscribed in the conic consisting of the two straight lines AE, BF, and having the two opposite sides AB, EF parallel. If then BD, CE are also parallel, AC, DF must be parallel, by Pascal's theorem.

As four forces present the simplest case of forces that may be in equilibrium without their lines of action meeting at a point, these curiously related quadrilaterals seem worthy of mention.

P. J. HEAWOOD.

76. *Note on a statement in Salmon's Conics.*

At the end of Art. 122 of Salmon's *Conics* (6th edition) there is a passage (faithfully reproduced in the German edition by the able Dr. Fiedler), a remark on which may serve didactic purposes. The equation $\alpha\gamma=k\beta\delta$ in perpendicular coordinates will represent a circle when $k=-1$, if the origin be within the quadrilateral bounded by $\alpha=0$, $\beta=0$, etc. But the author adds, that if the origin be outside then the condition is $k=+1$. Now it depends upon how the origin goes outside, if by merely crossing a side, then one perpendicular changes sign and the author's result is right: but if by passing through an angle (into the opposite vertical angle), then two perpendiculars change sign and the proper value of k is -1 as before.

R. W. GENESE.

77. *Note on Question* 309.

The axis of perspective envelopes a conic touching the sides of the triangle of reference.

The axis is $\Sigma \dfrac{\alpha}{p}=0$, ...(1)

where p, q, r are $\cos B \cos C+\cos A \cos 2\theta$, etc.

For envelope,

$$\Sigma \frac{\alpha \cos A}{p^2}=0\ ;\ \therefore\ \frac{\alpha/p}{\cos C/r-\cos B/q}=\text{etc.}\ ;\ i.e.\ \frac{\alpha}{p^2\cos A\,(\cos^2 B-\cos^2 C)}=\text{etc.}$$

whence $\alpha/[p^2\cos A \sin A \sin (B-C)]=\text{etc.}$

From (1) the envelope is $\Sigma \sqrt{\alpha \cos A \sin A \sin (B-C)}=0$,

which may be written $\Sigma \sqrt{\alpha (b^2-c^2) \cos A}=0$. J. F. HUDSON.

78. *Demonstrations of* Euc. XIII. 10 *on which depends the method used in Geometrical Drawing for inscribing a regular pentagon in a circle.*

Let EB, BD be respectively the sides of a regular in-pentagon and in-decagon in a circle, centre A. Draw DH, EK perpendicular to AB.

$$E\hat{A}B=2B\hat{A}D=A\hat{B}D\ ;\quad A\hat{C}E=B\hat{C}D\ ;\quad \therefore\ A\hat{E}C=B\hat{D}C=\tfrac{1}{2}E\hat{A}B=B\hat{A}D.$$

$\therefore$ triangles AEC, DEB are each equiangular to ABD and isosceles.

$\therefore$ DH, EK bisect the bases BC, CA ; and $A\hat{D}E=A\hat{E}D=C\hat{A}D$;

$\therefore$ $AC=CD=BD$.

Now $EB^2=2AB, BK$ (II. 13) $=AB(AB+BC)=AB^2+AB, BC$

$=AB^2+AC^2=AB^2+BD^2$,

or $EB^2=2AB, BK=2AB, BH+2AB, HK=BD^2+AB^2$ (II. 13).

These are modifications of proofs in Leslie, *Elements of Geometry* (4th ed., p. 121). E. M. LANGLEY.

PROBLEMS.

340. [K. 20. e.]. The Euler line of a triangle ABC meets BC at an angle such that $\tan \alpha(\tan B-\tan C)=3-\tan B \tan C$. Find the relation connecting the angles α, β, γ that the Euler line makes with the sides of the triangle.

E. N. BARISIEN.

341. [U.]. There was a new moon on Jan. 1, 1 B.C. (*Barlow and Bryan*, p. 215), and on Jan. 1, 1900. How many have there been on a Jan. 1st in that interval? Give a general formula. C. BICKERDIKE.

342. [I. 18. a. c.]. The sum or difference of two perfect cubes is, or has a factor, of the form $3m^2+n^2$. R. F. DAVIS.

343. [L^1. 7. d.]. If one focus of an in-conic lie on the Euler line of the triangle, find the locus of the second focus. J. M. DYER.

344. [L^1. 10. b.]. OP, OQ, OR are normals to a parabola, focus S, vertex A. Prove that $\Sigma A\hat{S}P=2n\pi+2A\hat{S}O$. R. W. GENESE.

345. [L^1. 2. c.]. Find geometrically the locus of the in-centres of triangles, vertices on three given lines, and equiangular with a given triangle.

J. F. HUDSON.

346. [K. 7. a.]. *AB*, *CD* intersect in *O*; *AB*, *CD*, *AC* cut the line at infinity in *L*, *M*, *P* respectively. *I*, *J* are the circules. Prove that

$$AB/CD = \{BAOL\}\{OCDM\}/\sqrt{\{LPMI\}\{LPMJ\}}.$$

W. J. JOHNSTON.

347. [P. 3. b.]. A variable circle, which inverts from each of two fixed points into circles of constant radii, envelopes a pair of circles.

C. E. M'VICKER.

348. [K. 9. a.] If a polygon be described having its sides equal, parallel, and in the same sense, but not in the same order, as the sides of a given closed polygon, the first-mentioned polygon is also closed. F. S. MACAULAY.

349. [J. 1. b.]. In how many ways may one row of n postage stamps be folded into a pile? NUMERATOR.

SOLUTIONS.

UNSOLVED QUESTIONS.—57, 129, 144, 152, 171, 175, 252, 271, 275, 279, 283, 285, 287, 306-8, 311-13, 320, 326-7, 334, 336-9.

Solutions of these questions, and of 340-359, should reach the Editor not later than April 15th. They will be published as space is available.

The question need not be re-written; the number should precede the solution. Figures should be very carefully drawn on a small scale, and on a separate sheet.

258. [I. 17. b. c.]. *Every number can be expressed as the sum of* 1, 2, 3, *or* 4 *squares. How many numbers are there less than* 1000, *which cannot be expressed as the sum of* 1, 2, *or* 3 *squares?* W. ALLEN WHITWORTH.

Solution by PROPOSER.

The required numbers are those of the forms

$$\alpha = 8N-1, \text{ or } \beta = P^2(8n-1).$$

But if P be odd, all the β's are included among the α's. And if P be the product of an odd and even factor, $P = OE$ suppose, then all the β's are included in the form $E^2(8n-1)$;

$\therefore$ we have to count the numbers of the forms

$$8N-1,\ 4(8N-1),\ 16(8N-1),\ 64(8N-1),$$

and it is at once seen that these forms give

$$125+31+7+2=165 \text{ numbers.}$$

[Among the numbers from 1 to 1000, 31 are $\square$, 299 are $\square+\square$, 505 are $\square+\square+\square$, 165 are $\square+\square+\square+\square$.]

275. [R. 4. c.] *Two light rods AB and AC, freely jointed at A, rest in a vertical plane; B and C are in contact with a smooth horizontal plane. Two other light rods DE and EF are rigidly connected at E, and hang with a heavy body supported at E, their ends D and F carrying smooth rings sliding on AB and AC respectively. Required the stress at A and the tensions in DE and EF.* St. John's (C), 1891.

Note by W. J. DOBBS.

The word "rigidly" appears to be an error, and as "tensions" of rods are required, I imagine they should be "freely jointed."

303. [L¹. 3. a. 11. c.]. *Find the locus of the centre of a rectangular hyperbola touching three straight lines; applying the principle that a circle whose ordinates are stretched in the ratio* 1 : ι *becomes a rectangular hyperbola.*

W. J. JOHNSTON.

Solution by PROPOSER.

$\Sigma l\alpha^2=0$ represents a rectangular hyperbola if $\Sigma l=0$, and is then satisfied by $\alpha=\pm\beta=\pm\gamma$. This result may be stated thus:

The centres of circles touching the sides lie on any rectangular hyperbola *w.r.t* which the given triangle is self-conjugate.

Take the transverse diameter of the hyperbola as axis, and stretch the ordinates in the ratio $1:\iota$. Any circle $x^2+y^2+2gx+2fy+c=0$ becomes $x^2-y^2+2gx+2f\frac{y}{\iota}+c=0$, viz., a rectangular hyperbola whose centre is the point corresponding to the centre of the circle.

The self-conjugate hyperbola becomes a circle.

Therefore the centres of rectangular hyperbolas touching the sides of a given triangle lie on the self-conjugate circle, which is the required locus.

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